Flag-manifold σ-model for SU(3) antiferromagnets on the triangular lattice & Néel-VBS transitions

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based on 2109.10051

$SU(N)$ quantum spins

Ordinary spins $\hat{S}_{x,y,z}$ with $\underbrace{[\hat{S}_a, \hat{S}_b] = i\,\varepsilon_{abc}\,\hat{S}_c}_{SU(2)\text{ algebra}}$

$\Rightarrow$ $SU(N)$ spins : Replace $SU(2)$ alg. to $SU(N)$ alg.

$$[\hat{S}_{i_1 i_2}, \hat{S}_{j_1 j_2}] = \delta_{i_1 j_2}\,\hat{S}_{i_2 j_1} - \delta_{i_2 j_1}\,\hat{S}_{i_1 j_2}$$

Why can it be interesting?

- Ultracold atoms (with alkaline earth metal) can realize it experimentally.
- $SU(3)$ spins can be realized by $SU(2)$ spin Hamiltonian (BBQ model) [Tsunetsugu, Arikawa 2006]
- (My motivation)

It gives new class of strongly-coupled $U(1)$ gauge theories in low dim!!

(Brief review of) $SU(2)$ antiferromagnets (AF) in D-dim.

Heisenberg model $\qquad \hat{H} = J \sum_{\langle i,j \rangle} \hat{S}(i) \cdot \hat{S}(j)$

$\uparrow \downarrow \uparrow \downarrow \uparrow \qquad SU(2)_{\text{spin}} \xrightarrow{\text{SSB}} U(1)$ (at least, classically)

Low-energy EFT : $d = (D+1)$-dim $\underbrace{\dfrac{SU(2)}{U(1)}}_{= \mathbb{CP}^1}$ relativistic σ-model.

<u>$D=1$</u> (spin chain) [Haldane, '83]

It accompanies the 2d topological θ-term with $\theta = 2\pi S$.

<u>$D=2$</u> (or $D \geq 3$) [Haldane, '88] [Read, Sachdev, '89, '90]

No topological terms. But, Berry phase gives interesting constraints on <u>monopoles</u>.

3d U(1) gauge theory for SU(2) AF in 2D square lattice.

EFT for Néel order (= SU(2) spin broken phase)

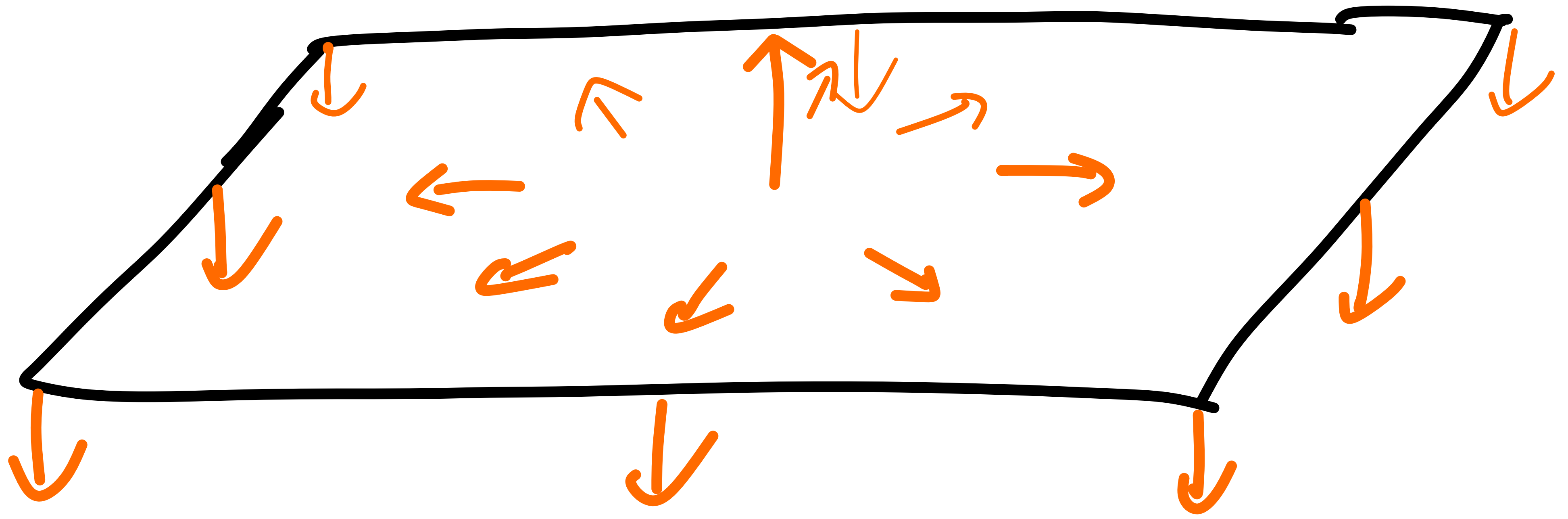
$$\mathcal{L} = \frac{1}{g^2}\int d^3x \, |(\partial_\mu + i a_\mu)\vec{\Phi}|^2.$$

← $\mathbb{CP}^1$ σ-model as U(1) gauge theory

($\vec{\Phi} \in \mathbb{C}^2$ with $|\vec{\Phi}|^2 = 1$, $a = a_\mu dx^\mu$: U(1) gauge field)

$\pi_2(\mathbb{CP}^1) = \mathbb{Z}$: Magnetic skyrmions

$$\boxed{Q_{skyrm} = \frac{1}{2\pi}\int_{xy} da}$$

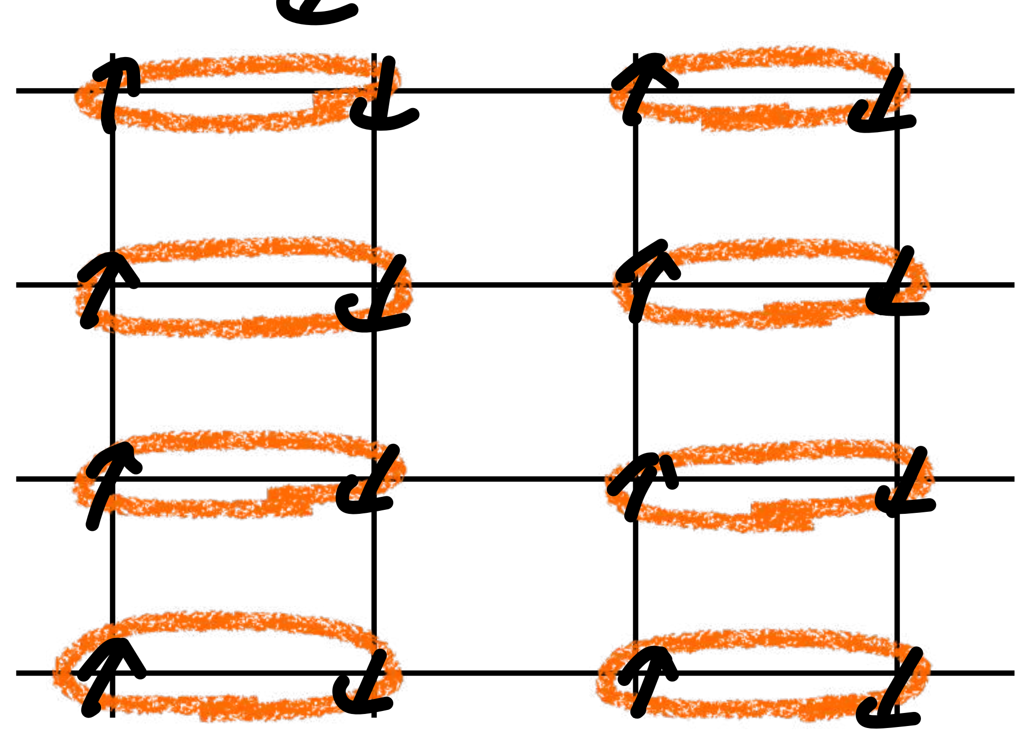


Monopole singularity $\mathcal{M}$

= Tunneling process for different skyrmion #

Néel – VBS transition for $SU(2)$

Another candidate for vacua of Heisenberg model:
Valence Bond Solid (VBS) phase



- $SU(2)$ spin ✓
- Lattice symmetry is broken. (For $2S \neq 0 \mod 4$)
  ($S=\frac{1}{2},\frac{3}{2} \Rightarrow$ 4 degenerate vacua, $S=1 \Rightarrow$ 2 degenerate)

We can regard this as monopole condensation phase of 3d $U(1)$ gauge theory. (Lattice symmetry $\Leftrightarrow$ Monopole symmetry $\mathcal{M} \to e^{i\alpha}\mathcal{M}$.)

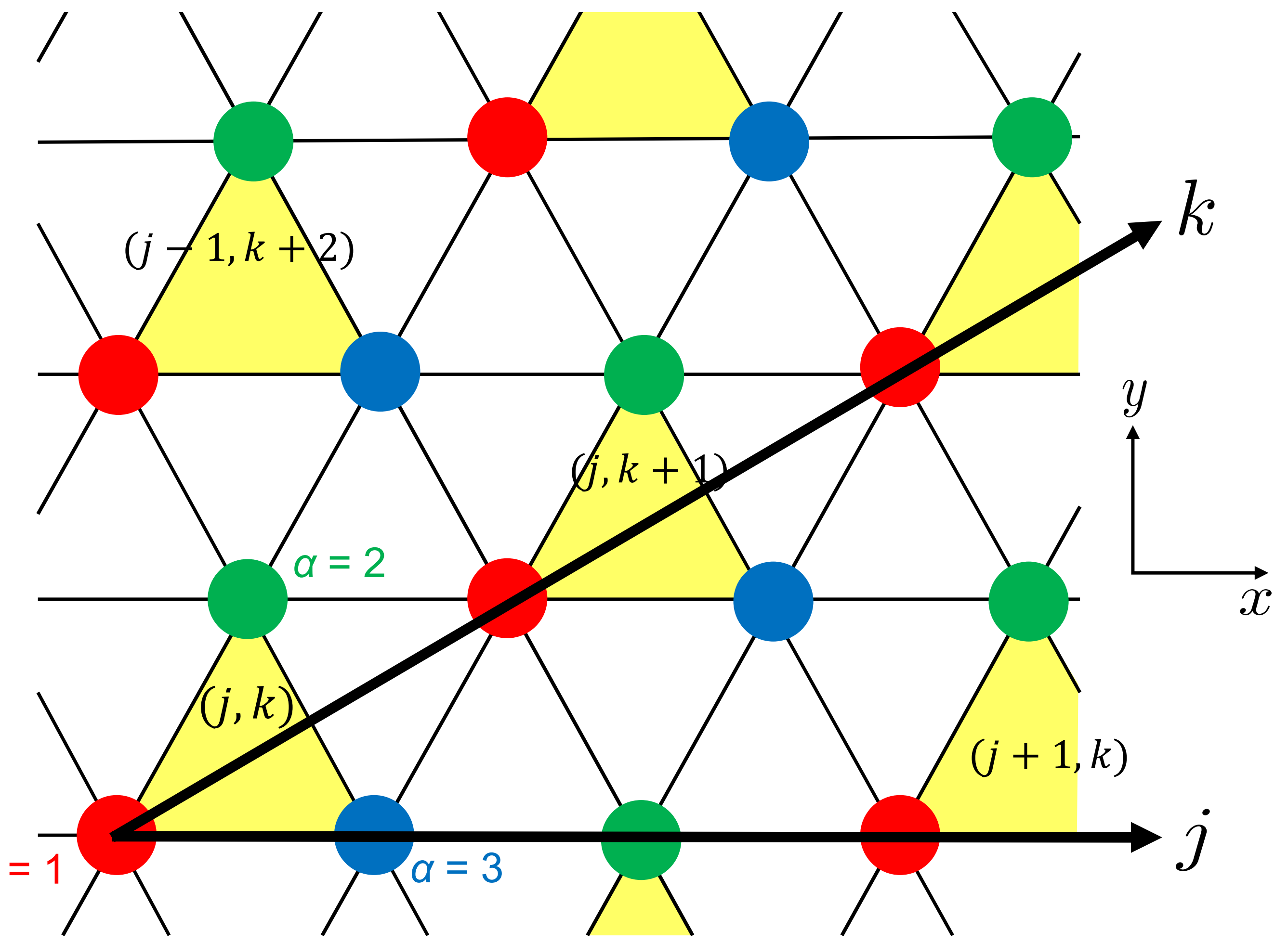
Direct quantum phase transition bet. Néel (Higgs) & VBS (Confinement) orders.

→ Deconfined quantum criticality (?) [Senthil et al. 2003]

Outline   SU(3) AF spins on the triangular lattice

$$H = J \sum_{\langle i,j \rangle} \mathrm{tr}\left(\underline{\hat{S}(i)}\ \underline{\hat{S}(j)} \right)$$

SU(3) spin
with p-box sym. rep. $\boxed{}\boxed{}\boxed{\cdots}\boxed{}$



- Classical Néel state $(p \to \infty)$: $SU(3)_{\mathrm{spin}} \xrightarrow{\mathrm{SSB}} U(1) \times U(1)$

$\Rightarrow$ EFT : 3d $\underbrace{\dfrac{SU(3)}{U(1)\times U(1)}}_{\text{flag manifold}}$ relativistic σ-model (No topological terms)

- Skyrmion has two U(1) charges as $\pi_2\left(\dfrac{SU(3)}{U(1)^2}\right) \simeq \mathbb{Z} \times \mathbb{Z}$.

Effect of dynamical monopoles : $\underline{\underline{\mathbb{Z}_3 \times \mathbb{Z}_3}} \subset U(1)^2$ $\quad (p \neq 0 \mod 3)$

characterising VBS

- 't Hooft anomaly matching

seems to be natural

Néel ——•—— VBS $\longrightarrow$ coupling   (or Néel ——•—— topological order ——•—— VBS)

Derivation of 3d flag-manifold σ-model

Schwinger boson approach

Decompose $SU(N)$ spin operators into N harmonic oscillators:

$$S_{ij} = a_i^\dagger a_j \qquad (i, j = 1, \cdots, N)$$

Constraint on the Hilbert space $\sum_i a_i^\dagger a_i = p$. # of boxes $\overbrace{\boxed{\ }\boxed{\ }\cdots\boxed{\ }}^{p}$

$$|\vec{\Phi}\rangle = \frac{1}{\sqrt{p!}} (\vec{\Phi} \cdot \vec{a}^\dagger)^p |0\rangle \quad (\vec{\Phi} \in \mathbb{C}^N, |\vec{\Phi}|^2 = 1)$$ gives

the coherent state:

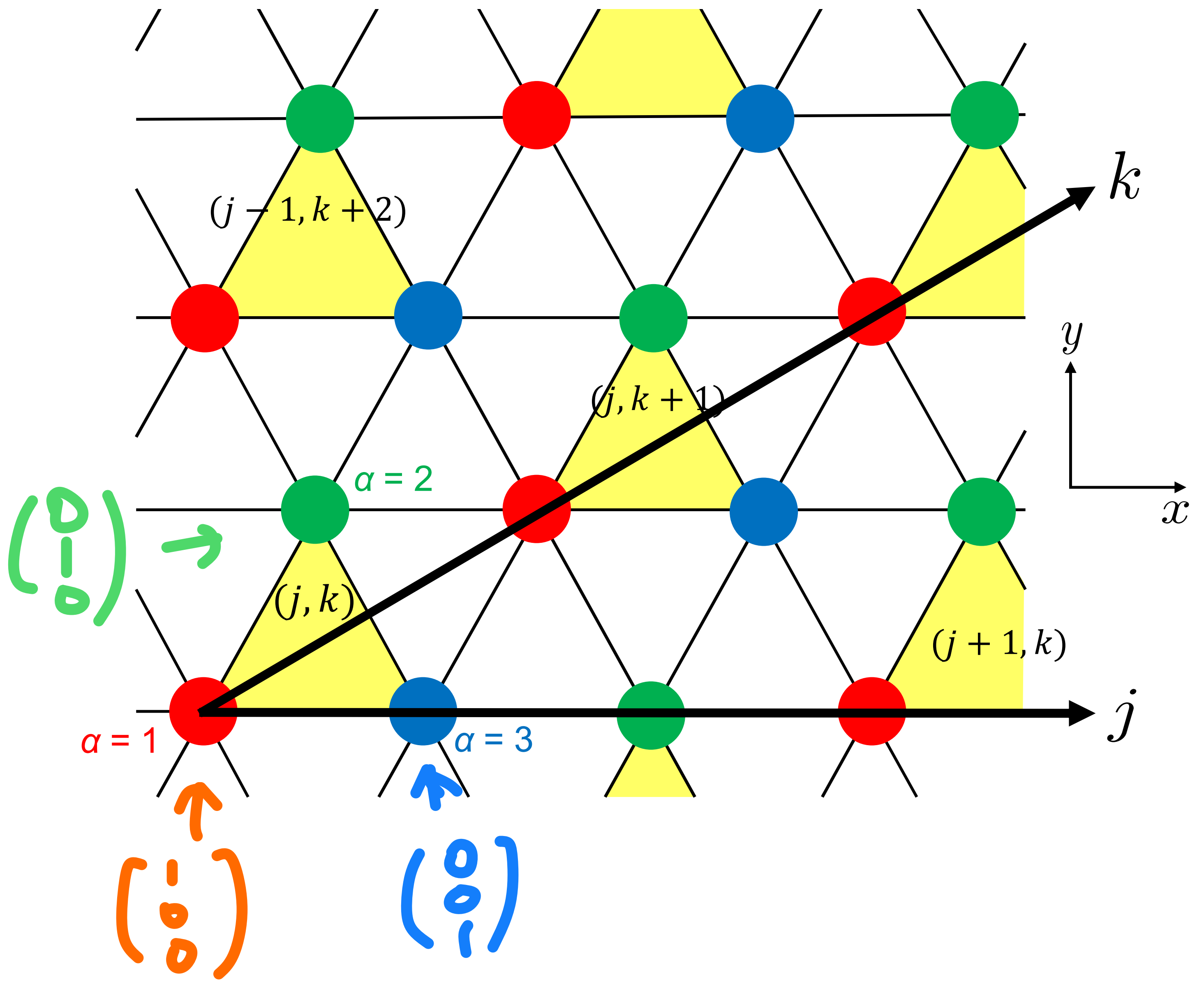
$$\langle \vec{\Phi} | S_{ij} | \vec{\Phi} \rangle = p \, \Phi_i^* \Phi_j .$$

This decomposition introduces $U(1)$ gauge redundancy.

Path integral of SU(3) spins

$$Z = \int \mathcal{D}\Phi \; \exp\left(-S[\Phi]\right)$$

$$S[\Phi] = \int_0^\beta d\tau \left\{ p \underbrace{\sum_i \vec{\Phi}^*(i)\,\partial_\tau \vec{\Phi}(i)}_{\text{Berry phase}} + \underbrace{J p^2 \sum_{\langle i,i' \rangle} \left| \vec{\Phi}^*(i) \cdot \vec{\Phi}(i') \right|^2}_{\text{classical energy}} \right\}$$



Lowest classical energy state

$$\Phi^*(i) \cdot \Phi(i') = 0 \quad \text{for } \langle i, i' \rangle .$$

RG transformation $\left(i \leftrightarrow (\underbrace{j,k}_{\text{coord.}}, \underbrace{\alpha}_{\text{sublattice}}) \right)$, with $(j,k) \to (x,y)$ and $\alpha = 1,2,3$

① Separate high- and low-energy fluctuations :

$$\underbrace{\left[\vec{\Phi}_1, \vec{\Phi}_2, \vec{\Phi}_3\right](j,k)}_{\text{three } \mathbb{CP}^2 \text{ vectors}} = \underbrace{L(j,k)}_{L = L^\dagger \text{ (high energy)}} \cdot \underbrace{U(j,k)}_{U \cdot U^\dagger = 1 \text{ (low energy)}}$$

② Integrate out $L(j,k)$.

EFT is 3d $\frac{SU(3)}{U(1)^2}$ σ-model

Denoting $U = [\vec{\phi}_1, \vec{\phi}_2, \vec{\phi}_3]$ $\quad (\vec{\phi}^*_\alpha \cdot \vec{\phi}_\beta = \delta_{\alpha\beta})$, we obtain

$$S = \int_0^\beta d\tau \int dx dy \, \frac{1}{g_{eff}} \sum_{\alpha=1}^{3} \left(\frac{1}{v} \left| (\partial_\tau + i a_{\alpha,\tau}) \vec{\phi}_\alpha \right|^2 + v \sum_{I=x,y} \left| (\partial_I + i a_{\alpha,I}) \vec{\phi}_\alpha \right|^2 \right)$$

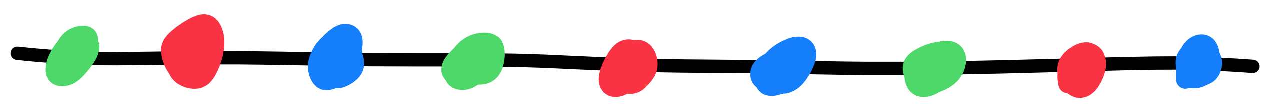
$$\left(g_{eff} = \frac{3\sqrt{3}}{\sqrt{2}} \cdot \frac{a}{p} \right.$$ lattice const. $$, \quad \left. v = \frac{3}{\sqrt{2}} J a p \right)$$ [See also Smerald, Shannon, 2013]

By rescaling $\tau \to \frac{\tau}{v}$, we can make this action relativistic inv. with $v = 1$.

Note : No topological term appears in this model.

Absence of topological terms

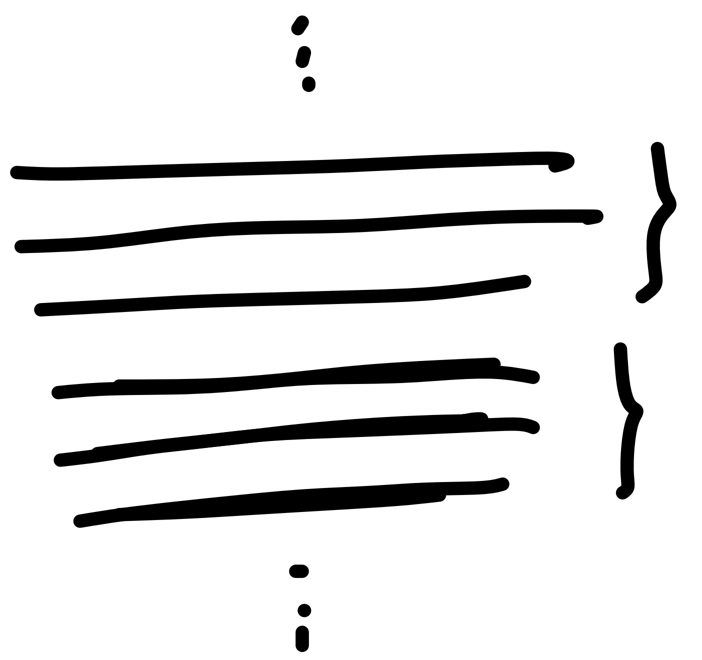
SU(3) AF spin chain $\longleftrightarrow$ 2d $\frac{SU(3)}{U(1)^2}$ σ-model with



$$(\theta_1, \theta_2) = \left(\frac{2\pi}{3}p, \frac{4\pi}{3}p\right)$$

[Bykov, '12, '13
Lajko, Wamer, Mila, Affleck, '17]

2D SU(3) AF



stack 3 spin chains

$(\theta_1,\theta_2) \sim (0,0)$ mod $2\pi\mathbb{Z}\times 2\pi\mathbb{Z}$ $\Rightarrow$ No topological terms appear.

(*) Relativistic 3d $\frac{SU(3)}{U(1)^2}$ σ-model does not have any θ terms as $\Omega_3^{\mathrm{Spin}}\left(\frac{SU(3)}{U(1)^2}\right) = 0$.

It can have WZ term $\left(\Omega_4^{\mathrm{Spin}}\left(\frac{SU(3)}{U(1)^2}\right) \simeq (2\mathbb{Z})^{\oplus 2}\right)$, but the underlying lattice symmetry has to be explicitly broken. [Kobayashi, Lee, Shiozaki, YT, 2103.05035]

Berry phase of monopoles

Skyrmion numbers & monopoles

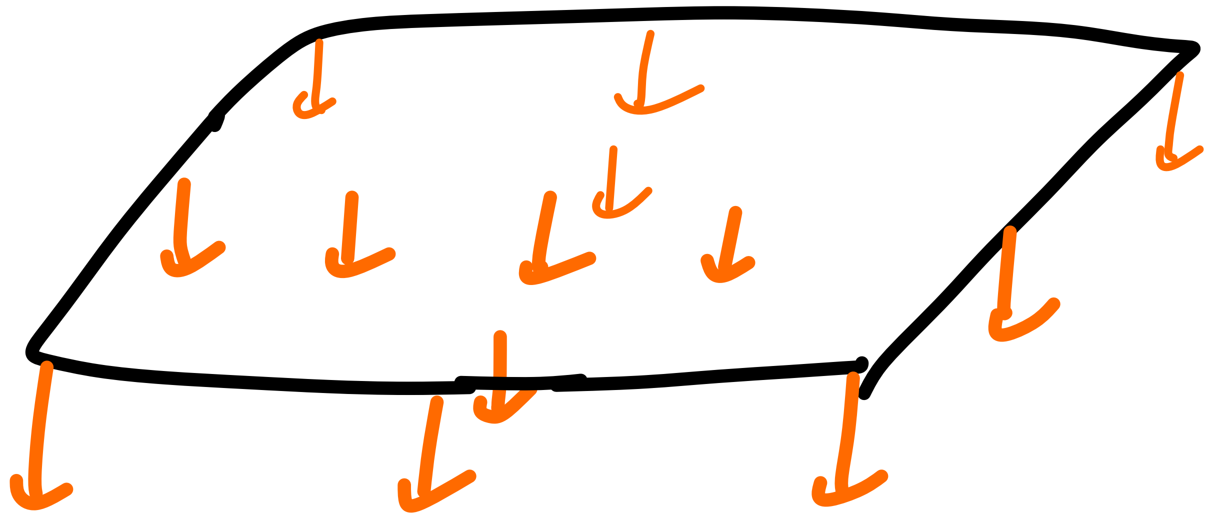
$$\pi_2\left(\frac{SU(3)}{U(1)^2}\right) = \mathbb{Z}\times\mathbb{Z} . \quad \Longleftarrow \text{ Topological conservation law.}$$

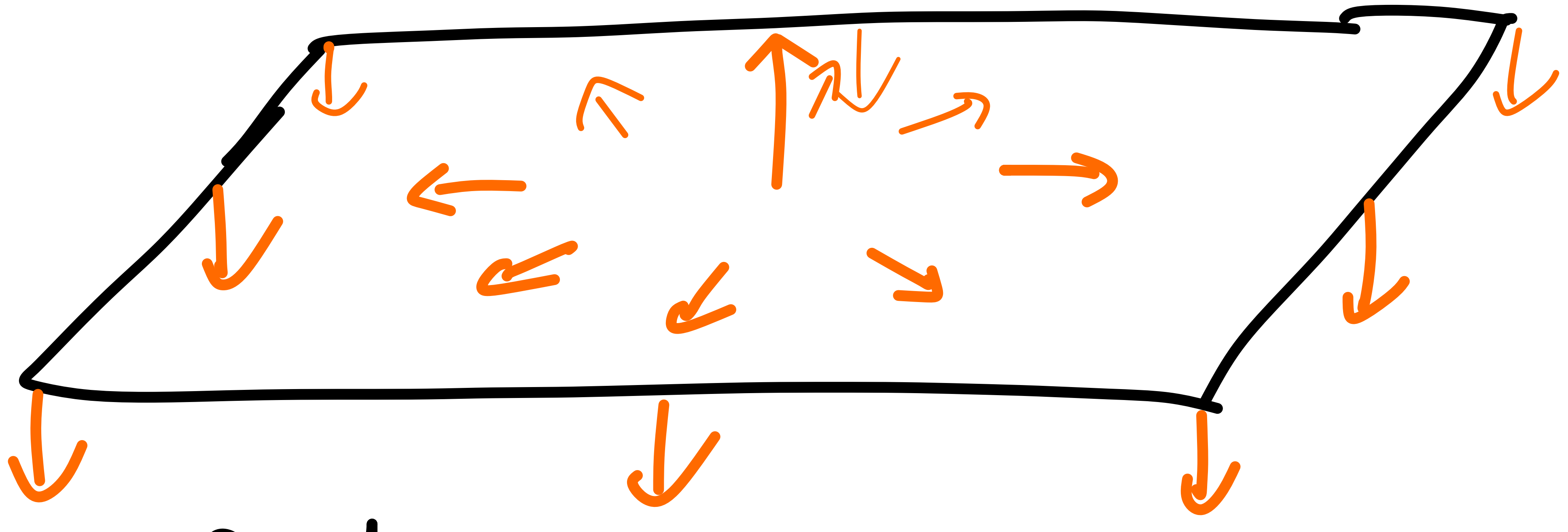
$$j_1 = \frac{1}{2\pi} da_1 , \quad j_2 = \frac{1}{2\pi} da_2 .$$

($*$ $j_3 = \frac{1}{2\pi} da_3$ conserves
but $j_1 + j_2 + j_3 = 0$)

Note that this is an accidental symmetry of continuum description.

At the lattice scale, it is explicitly broken ($\leftrightarrow$ dynamical monopoles)





$$Q = \int_{xy} \frac{da}{2\pi} = 0$$

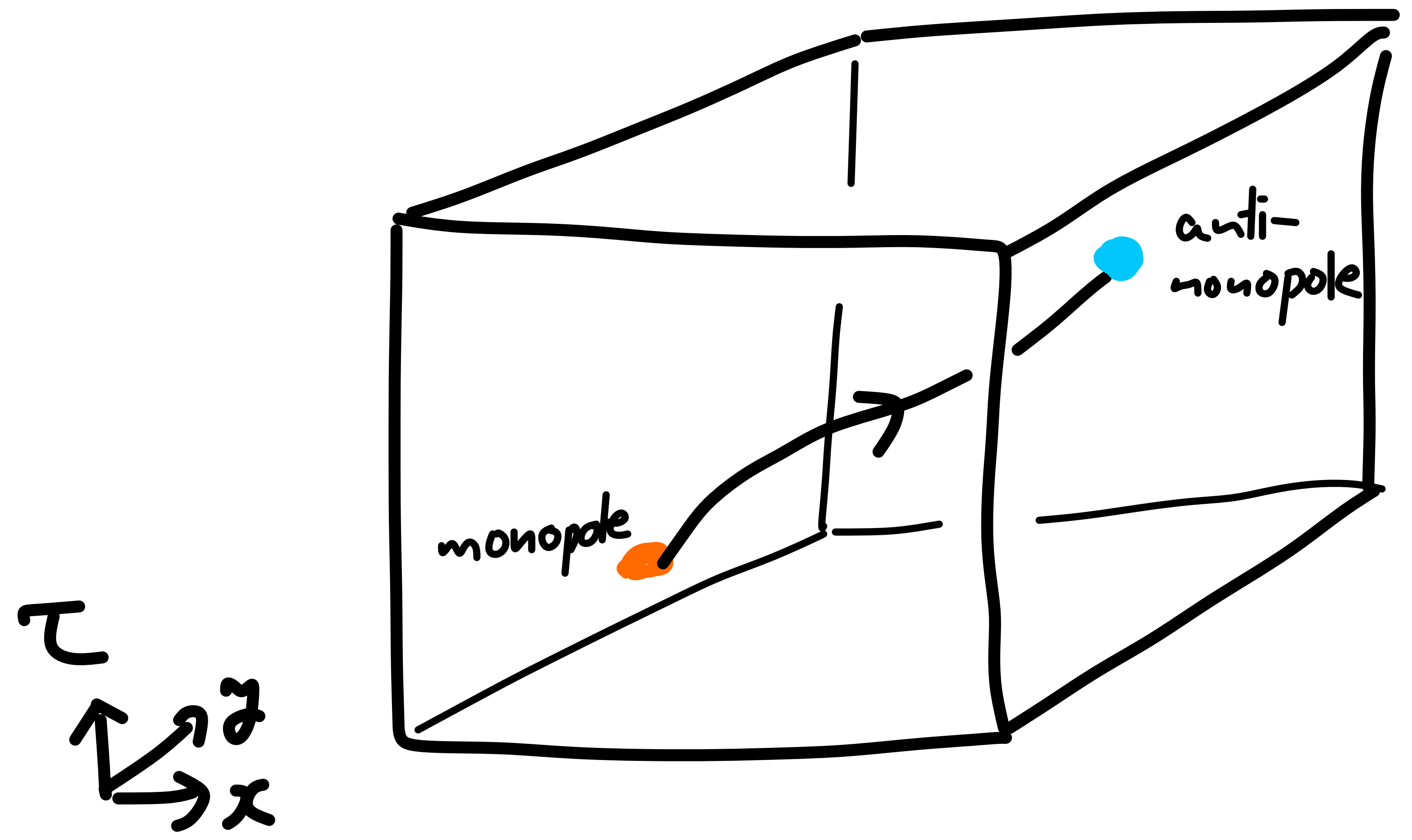
$$Q = \int_{xy} \frac{da}{2\pi} = 1$$

Berry phase revisited

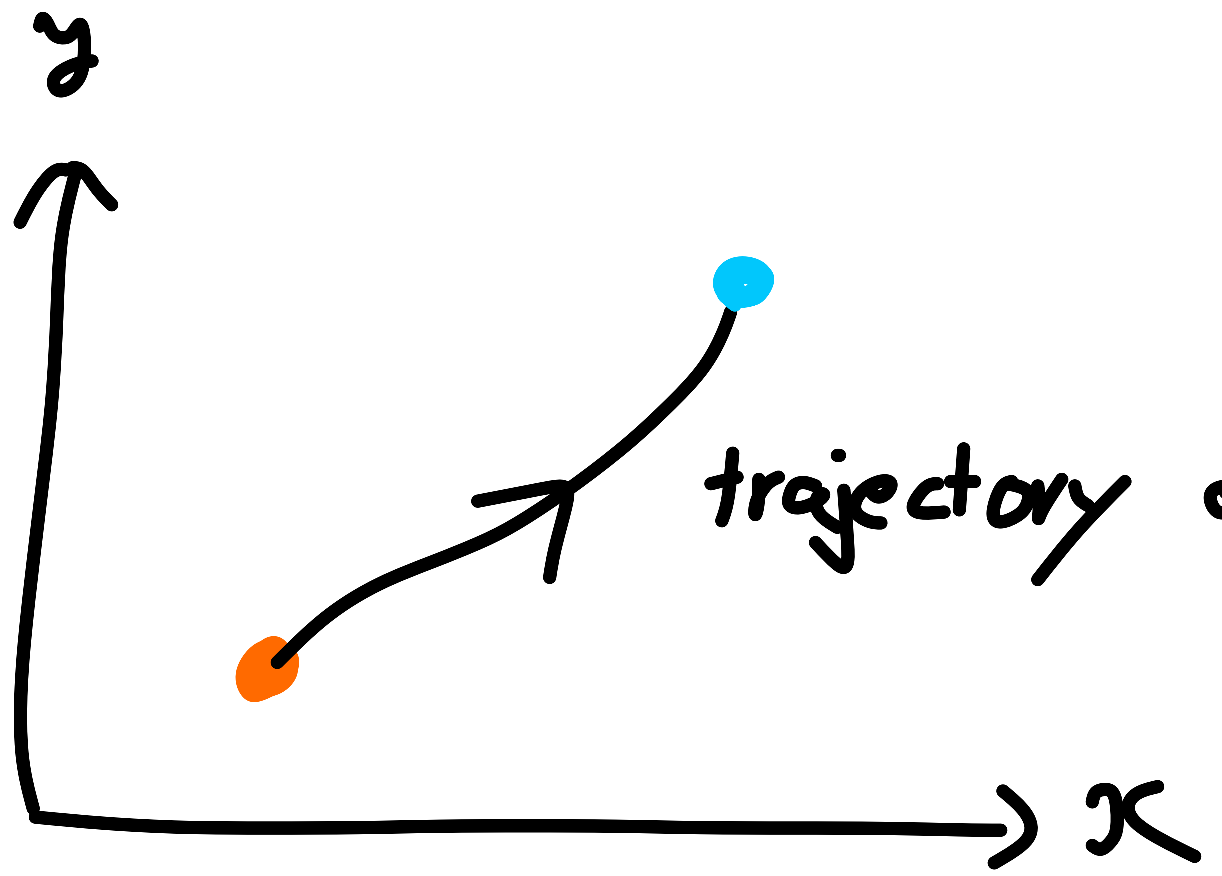
$S_B = p \sum_i \int \Phi^*(i)\, \partial_\tau \Phi(i)\, d\tau$ can be complex.

But, when continuum approx. is valid, no topological term appears.

This is not correct at the monopole events.



projection to xy $\longrightarrow$



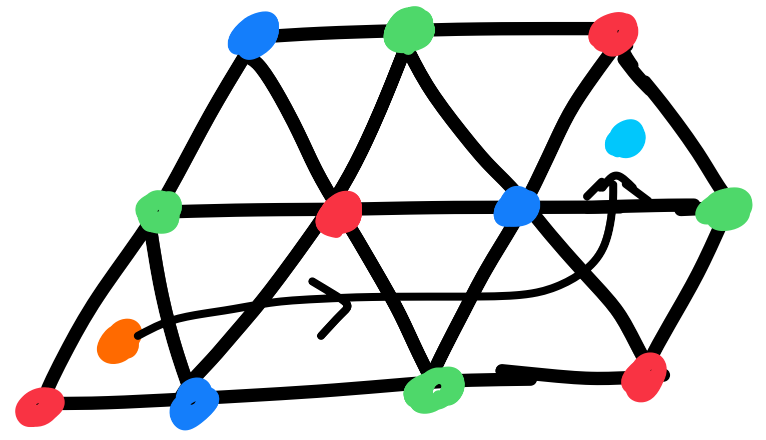
$\Downarrow$

discontinuity line of the Berry phase [Haldane, '88]

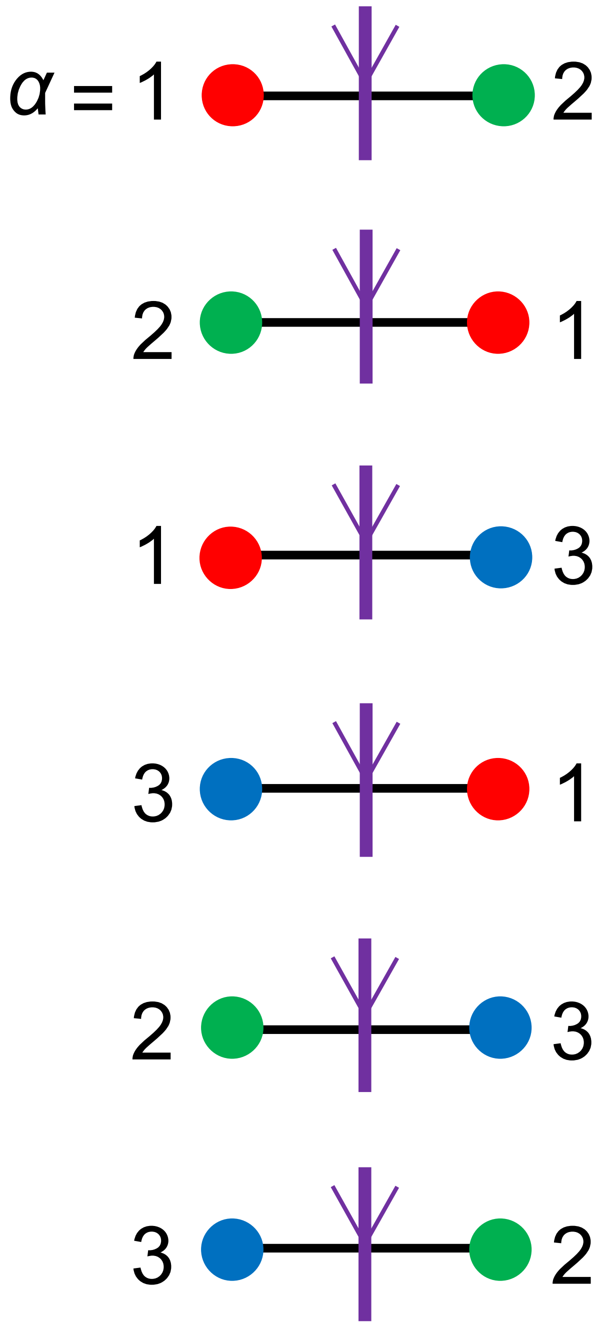
Assumption : Skyrmions are created/annihilated at the center of △ lattice.

$\Downarrow$

We find a nice graphical rule for S_B.



$$S_B = \sum_{\substack{\text{edges} \\ \text{cut by discontinuity line}}} \Delta S_B$$



$$\Delta S_{\mathrm{B}} = ip\left(-\frac{2\pi}{9}Q_1 + \frac{2\pi}{9}Q_2\right)$$

$$\Delta S_{\mathrm{B}} = ip\left(+\frac{2\pi}{9}Q_1 - \frac{2\pi}{9}Q_2\right)$$

$$\Delta S_{\mathrm{B}} = ip\left(-\frac{4\pi}{9}Q_1 - \frac{2\pi}{9}Q_2\right)$$

$$\Delta S_{\mathrm{B}} = ip\left(+\frac{4\pi}{9}Q_1 + \frac{2\pi}{9}Q_2\right)$$

$$\Delta S_{\mathrm{B}} = ip\left(-\frac{2\pi}{9}Q_1 - \frac{4\pi}{9}Q_2\right)$$

$$\Delta S_{\mathrm{B}} = ip\left(+\frac{2\pi}{9}Q_1 + \frac{4\pi}{9}Q_2\right)$$

(* Continuous deformation of discontinuity line does not change S_B.)

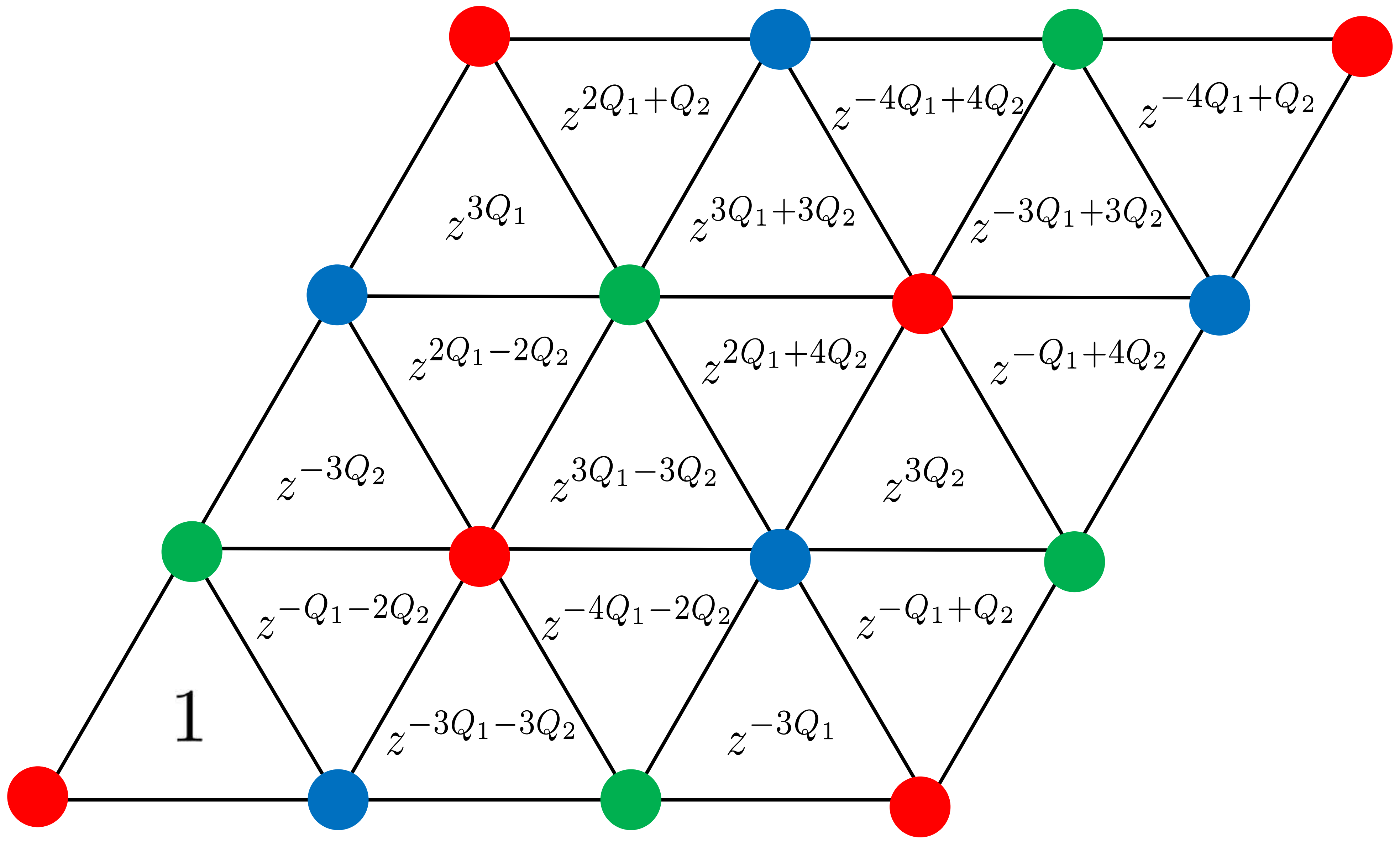
$\Rightarrow$ S_B depends only on the location of monopoles

Destructive interference

S_B gives the complex phase to the monopole operator $\mathcal{M} \sim e^{iQ_1\sigma_1 + iQ_2\sigma_2}$ depending on its location.

$$\left(z = e^{\frac{2\pi i}{9}} \right).$$

($p=1$ in the following)



In continuum, these phases should be averaged:

$$\underbrace{(1+z^{3Q_1}+z^{-3Q_1})}_{=0}\underbrace{(1+z^{3Q_2}+z^{-3Q_2})}_{=0}(1+z^{-Q_1-2Q_2}).$$

except $Q_1 \in 3\mathbb{Z}$.   except $Q_2 \in 3\mathbb{Z}$.

$\Rightarrow$ $\mathbb{Z}_3 \times \mathbb{Z}_3 \subset U(1)^2_{\text{top}}$ is a good symmetry.

VBS and monopole gas

If Néel order is destroyed by strong quantum fluctuations, monopoles are liberated. $\Rightarrow$ Confinement [Polyakov '77] $\quad (da_\alpha \sim * d\sigma_\alpha)$

$$V_{eff}^{(\text{monopole})} \propto -\Big(\cos(3\sigma_1) + \cos\left(3\sigma_1 + \frac{2\pi}{3}\right) + \cos(3\sigma_2) + \cos\left(3\sigma_2 - \frac{2\pi}{3}\right) + \cos(3(\sigma_1-\sigma_2)) + \cos\left(3(\sigma_1-\sigma_2) + \frac{2\pi}{3}\right)\Big).$$

<u>Vacua</u> $\quad (\sigma_1, \sigma_2) = \left(\frac{2\pi}{3} n_1, \frac{2\pi}{3} n_2\right), \left(\frac{2\pi}{3} n_1 - \frac{2\pi}{9}, \frac{2\pi}{3} n_2 + \frac{2\pi}{9}\right).$

$\Rightarrow$ 18 degenerate vacua by SSB of $\mathbb{Z}_6 \times \mathbb{Z}_3$ $(\supset \mathbb{Z}_3 \times \mathbb{Z}_3)$.

(* For VBS, we expect 6 degenerate vacua, so there is a factor 3 difference. We are now trying to fix this.)

Anomaly matching & phase diagram

't Hooft anomaly

3d $\frac{SU(3)}{U(1)^2}$ σ- model has an 't Hooft anomaly for

$$\left(\frac{SU(3)}{\mathbb{Z}_3}\right)_{\text{spin}} \times (\mathbb{Z}_3 \times \mathbb{Z}_3)_{\text{top}}. \qquad [\text{YT, Sulejmanpasic, '17}]$$

A : SU(3) 1-form

B : $\mathbb{Z}_3$ 2-form

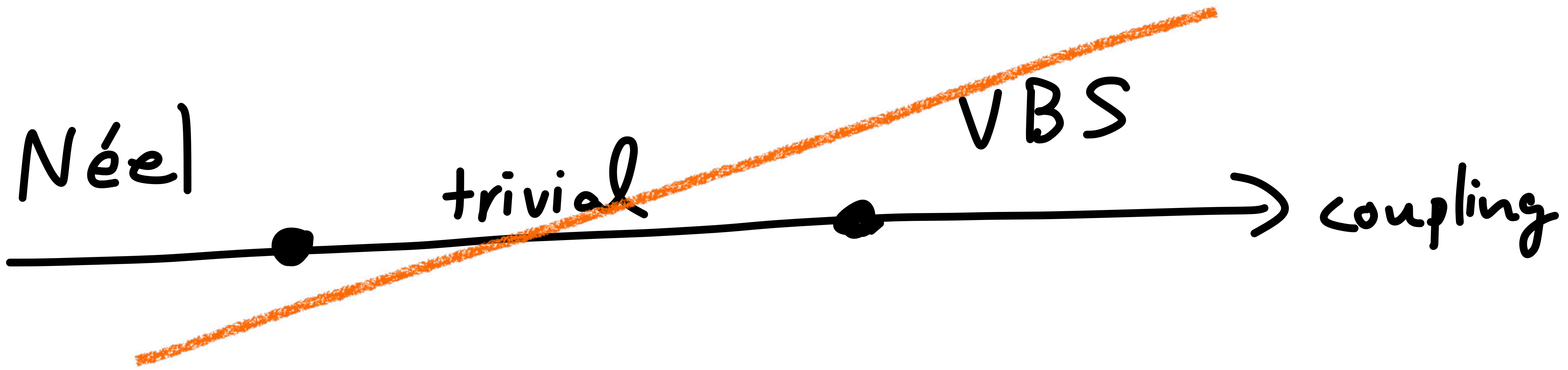
A_1, A_2

Anomaly is given by the boundary of the 4d SPT action,

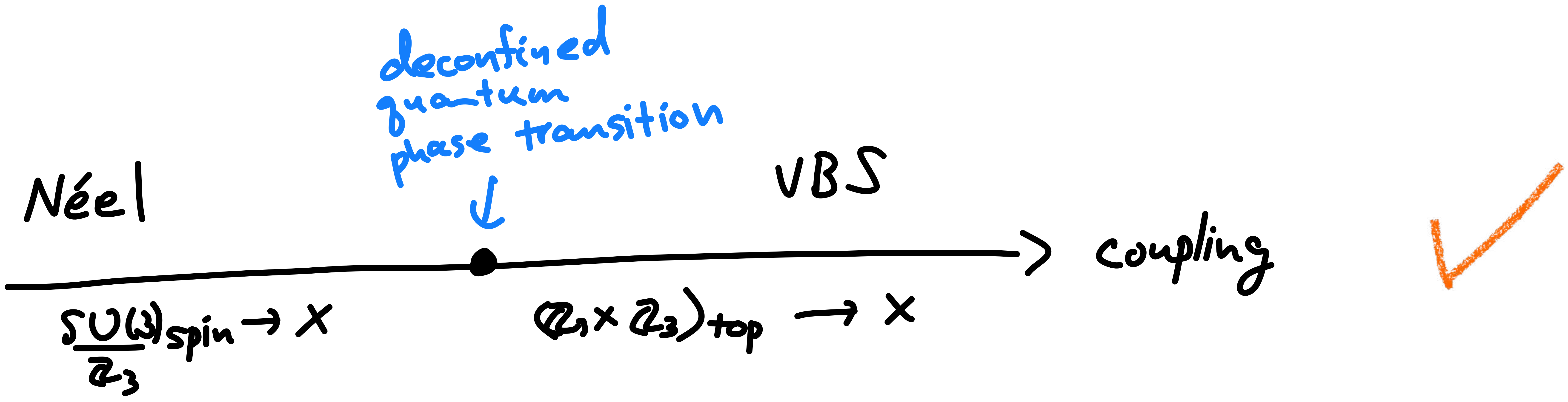
$$S_{4d\ SPT} = \frac{1}{2\pi}\int_{4d} (dA_1 + dA_2) \wedge B \in \mathbb{Z}_3.$$

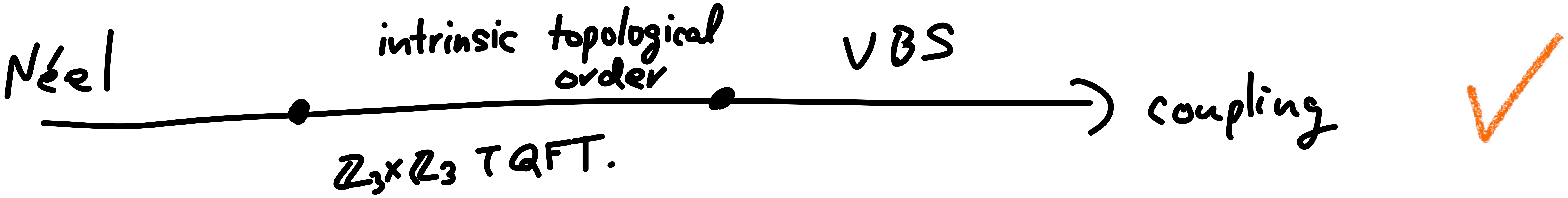
(* Full symmetry structure is not identified yet. It should be more complicated. Still, this information is interesting enough to constrain the phase structure.)

Anomaly matching constraint on the phase diagram



This looks to be most reasonable for us. $\Rightarrow$





Summary

- 3d $\frac{SU(3)}{U(1)^2}$ σ-model without topological terms describe the 2D SU(3) AF spins on $\triangle$ lattice.

- We compute the Berry phase of monopoles.

  $\Rightarrow$ Destructive interference is found, and we identify the monopole symmetry & its SSB.

  (Mismatch of degeneracy ?)

- Anomaly matching suggests the direct phase transition for Néel & VBS.

  (Full structure of symmetry ?)